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THE STRICT TOPOLOGY FOR DOUBLE CENTRALIZER ALGEBRAS(1)

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Abstract. Sufficient conditions are given for a double centralizer algebra under the strict topology to be a Mackey space.

0. **Introduction.** Let $C(S)$ be the B^*-algebra of all bounded complex valued continuous functions on a locally compact Hausdorff space S; let $C_0(S)$ be the algebra of all functions in $C(S)$ that vanish at infinity, and let $C(S)_\beta$ denote $C(S)$ under the β or strict topology. In 1958, R. C. Buck [3] proved that the strict dual of $C(S)$ under the strong topology is isometrically isomorphic to the norm dual of $C_0(S)$ and then raised the following question: Is it in fact true that the strict topology β coincides with the Mackey topology? In 1967, J. B. Conway [6] answered this question for the most part. He showed that if S is paracompact, then indeed the strict topology is the Mackey topology and he also gave examples of locally compact spaces S where the strict topology for $C(S)$ is not the Mackey topology.

More recently, R. C. Busby [4] in his study of double centralizers of B^*-algebras introduced a generalized notion of the strict topology. Specifically, if A is a B^*-algebra and $M(A)$ is its double centralizer algebra, then the strict topology β for $M(A)$ is defined to be that locally convex topology generated by the seminorms $(\lambda_a)_{a\in A}$ and $(\rho_a)_{a\in A}$, where $\lambda_a(x)=\|ax\|$ and $\rho_a(x)=\|xa\|$, and we let $M(A)_\beta$ denote $M(A)$ under the strict topology. Although Busby investigated some of the properties of the strict topology in this setting, no mention was made of the strict dual of $M(A)$. Thus, the questions under consideration are the following: (1) Is the strict dual of $M(A)$ under the strong topology a Banach space that is isometrically isomorphic to the norm dual of A? (2) What are some sufficient conditions for the strict topology for $M(A)$ to be the Mackey topology? The answer to question (1) is yes and to answer question (2) we prove the following two theorems:

THEOREM I. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of B^*-algebras and let $A=(\sum A_\lambda)_0$. Then $M(A)_\beta$ is a Mackey space if, and only if, for each $\lambda\in\Lambda$, $M(A_\lambda)_\beta$ is a Mackey space.*

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THEOREM II. *Let A be a B^*-algebra and suppose one of the following conditions holds*:

(1) *$M(A)$ is isometrically $*$-isomorphic to the bidual of A.*

(2) *A has a countable approximate identity.*

Then $M(A)_\beta$ is a Mackey space.

If S is a locally compact paracompact Hausdorff space, then by [2, p. 107] S can be expressed as the union of a collection $\{Y_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint open and closed σ-compact subsets of S. For each $\lambda \in \Lambda$ set $A_\lambda = C_0(Y_\lambda)$ and observe that A_λ has a countable approximate identity. Since A and $M(A)$ are isometrically $*$-isomorphic to $C_0(S)$ and $C(S)$ respectively, where $A = (\sum A_\lambda)_0$, it follows that Theorem II, together with Theorem I, generalizes Conway's result [6, Theorem 2.6, p. 478] as well as a result of LeCam [11, Proposition 3, p. 220].

Furthermore Theorem II, together with the fact that the strict dual of $M(A)$ under the strong topology is isometrically isomorphic to the norm dual of A, gives for a special case a characterization of the Mackey topology of W^*-algebras (see [1]).

1. **Notation and preliminaries.** Let A be a B^*-algebra. By a double centralizer on A, we mean a pair (R, S) of functions from A to A such that $aR(b) = S(a)b$ for a, b in A, and we will denote the set of all double centralizers on A by $M(A)$. If $(R, S) \in M(A)$, then R and S are continuous linear operators on A and $\|R\| = \|S\|$, so $M(A)$ under the usual operations of addition and multiplication is a Banach algebra, where $\|(R, S)\| = \|R\|$. Furthermore, if we define $(R, S)^* = (S^*, R^*)$, where $R^*(a) = (R(a^*))^*$ and $S^*(a) = (S(a^*))^*$ for all $a \in A$, then $(R, S)^* \in M(A)$ and this implies that $M(A)$ is a B^*-algebra. If we define a map $\mu_0 : A \to M(A)$ by the formula $\mu_0(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$ for all $b \in A$, then μ_0 is an isometric $*$-isomorphism from A into $M(A)$ and $\mu_0(A)$ is a closed two sided ideal in $M(A)$. Hence throughout this paper we will view A as a closed two sided ideal in $M(A)$. If A is commutative, then $M(A)$ is isometrically $*$-isomorphic to the algebra of multipliers as studied by Wang [17]. If $\{A_\lambda\}$ is a family of B^*-algebras, then $\sum A_\lambda$ and $(\sum A_\lambda)_0$ are defined as in [12]. It is clear that $\sum A_\lambda$ and $(\sum A_\lambda)_0$ are B^*-algebras. For a more detailed account of the theory of double centralizers on a B^*-algebra, we refer the reader to [4], and for definitions and concepts in general, we refer the reader to [10] and [12].

2. **The dual of $M(A)_\beta$.** In this section we prove that the strict dual of $M(A)$ under the strong topology is isometrically isomorphic to the norm dual of A and furthermore, we characterize the β-equicontinuous subsets of the strict dual of $M(A)$.

THEOREM 2.1. *Let A be a B^*-algebra and let A^* denote the dual of A. Then $A^* = \{a \cdot f : a \in A$ and $f \in A^*\} = \{f \cdot a : a \in A$ and $f \in A^*\}$, where $a \cdot f(b) = f(ba)$ and $f \cdot a(b) = f(ab)$ for all $b \in A$.*

Proof. Let f be a positive linear functional in A^*. By virtue of [12, Theorem 4.5.14, p. 219] f is representable; that is, there exists a Hilbert space H, a continuous $*$-representation $a \to T_a$ of A on H, and a topologically cyclic vector h_0 in H such that $f(a)=(T_a h_0, h_0)$ for all $a \in A$. Let $\{e_\lambda\}$ be an approximate identity for A. Since $h_0=\lim T_{a_n} h_0$ for some sequence $\{a_n\}$ of elements in A, we can easily show that $\lim T_{e_\lambda} h_0 = h_0$. Due to the fact that H is an A-module in the sense of [9, Definition 2.1, p. 147], we have by the Cohen-Hewitt factorization theorem [9, Theorem 2.5, p. 151] that $h_0 = T_a h_1$ for some $a \in A$ and $h_1 \in H$. Define g on A by the formula $g(b)=(T_b h_1, h_1)$ for each $b \in A$ and note that $g \in A^*$ and $f=a\cdot g \cdot a^*$.

Now assume that f is any element of A^*. Since f can be expressed as a finite linear combination of positive functionals on A [14, Theorem 1, p. 439], we see that $\lim e_\lambda \cdot f = \lim f \cdot e_\lambda = f$. Hence, by [9, Theorem 2.5, p. 151], there exist elements a and b in A and linear functionals g_1 and g_2 in A^* such that $f=a\cdot g_1 = g_2 \cdot b$ and our proof is complete.

COROLLARY 2.2. *If A is a B^*-algebra, then $M(A)^*_\beta = \{a\cdot f : a \in A$ and $f \in M(A)^*\} = \{f\cdot a : a \in A$ and $f \in M(A)^*\}$, where $a\cdot f(x)=f(xa)$ and $f\cdot a(x) = f(ax)$ for all $x \in M(A)$.*

Proof. Due to the fact that the strict topology is weaker than the norm topology, we have that $M(A)^*_\beta \subset M(A)^*$. Now let $f \in M(A)^*_\beta$ and let ϕf denote the restriction of f to A. By Theorem 2.1 there exists an $a \in A$ and a $g \in A^*$ such that $\phi f = a \cdot g$. By the Hahn-Banach theorem there exists an $h \in M(A)^*$ such that $g = \phi h$. Now let $\{e_\lambda\}$ be an approximate identity for A and let $x \in M(A)$. Since $e_\lambda x + x e_\lambda - e_\lambda x e_\lambda$ converges to x in the strict topology and A is a closed two sided ideal in $M(A)$, we have that

$$\begin{aligned} f(x) &= \lim f(e_\lambda x + x e_\lambda - e_\lambda x e_\lambda) = \lim a\cdot g(e_\lambda x + x e_\lambda - e_\lambda x e_\lambda) \\ &= g(xa) = h(xa) = a \cdot h(x). \end{aligned}$$

Hence $f = a\cdot h$ and similarly there is a $b \in A$ and an $h_1 \in M(A)^*$ such that $f = h_1 \cdot b$. Since it is easy to show that $a\cdot f$ and $f \cdot a$ are strictly continuous for each $a \in A$ and $f \in M(A)^*$, our proof is complete.

The strong topology for $M(A)^*_\beta$ is defined to be the topology of uniform convergence on the β-bounded subsets of $M(A)_\beta$.

COROLLARY 2.3. *If A is a B^*-algebra, then $M(A)^*_\beta$ under the strong topology is a Banach space that is isometrically isomorphic to A^*.*

Proof. By virtue of the uniform boundedness principle, it is straightforward to show that the β-bounded subsets of $M(A)$ are norm bounded. Therefore, the strong topology for $M(A)^*_\beta$ is the usual topology generated by the norm of $M(A)^*$. Since A is strictly dense in $M(A)_\beta$, we have by Theorem 2.1 and Corollary 2.2 that the restriction map ϕ is an isomorphism of $M(A)^*_\beta$ onto A^*. Therefore, to complete the proof we need to show that ϕ is an isometry. But this follows from the fact that $f(x)=\lim f(xe_\lambda)$ for each $f \in M(A)^*_\beta$ and $x \in M(A)$, where $\{e_\lambda\}$ is an approximate identity for A.

LEMMA 2.4. *Let A be a B^*-algebra and let $\{d_n\}$ be a sequence of elements of A, $\|d_n\|<1$, that converges to zero. Then there exist sequences $\{b_n\}$ and $\{c_n\}$ of elements of A and a hermitian element a of A, $\|a\| \leqq 1$, such that*

(1) $d_n=ab_n=c_na$;

(2) $\|d_n\| \geqq \max\{\|b_n\|^2, \|c_n\|^2\}$.

Proof. Let A_1 be the B^*-algebra obtained by adjoining the identity, let $\{e_\lambda\}$ be an approximate identity for A consisting of hermitian elements, and let $Z=\{x \in A : x=d_n,\ x=d_n^*,\ x=(d_nd_n^*)^{1/4}$, or $x=(d_n^*d_n)^{1/4}\}$. Since $e_\lambda x \to x$ uniformly on Z, we may define by induction a sequence $\{e_{\lambda_k}\}$ of elements in the unit ball of A such that $\|x-e_{\lambda_n}x\|<\delta/8^{n+1}$, $x \in Z$, and $\|e_{\lambda_k}-e_{\lambda_{n+1}}e_{\lambda_k}\|<\delta/32^{n+1}$, $k=1, 2, \ldots, n$, where $\delta=\min\{1-\|d_n\|^{1/2} : n=1, 2, 3, \ldots\}$. Now set

$$a_n = \sum_{k=1}^{n} \nu(1-\nu)^{k-1}e_{\lambda_k}+(1-\nu)^n, \quad \text{where } 0 < \nu < 1/4.$$

It follows, as in the proof of [16, Theorem 2.1], that a_n^{-1} exists, $\|a_n^{-1}\| \leqq 4^n$, and $a_{n+1}^{-1}-a_n^{-1}=r(1-e_{\lambda_{n+1}})+s$, where $\|r\| \leqq 4^n$ and $\|s\| \leqq \delta/2^{n+2}$. These facts together with the fact that a_n^{-1} is hermitian gives us, as in the proof of [16, Theorem 2.1], that $\lim a_n^{-1}x$ and $\lim xa_n^{-1}$ exist for each $x \in Z$ and that $\|\lim a_n^{-1}x\| \leqq \|x\|+\delta$. So, by setting $b_n=\lim_{p\to\infty} a_p^{-1}d_n$, $c_n=\lim_{p\to\infty} d_na_p^{-1}$, and $a=\lim a_p$, we see that (1) holds. We now wish to show that (2) holds. But

$$\begin{aligned}\|b_n\|^2 &= \|b_nb_n^*\| = \lim_{p\to\infty} \|a_p^{-1}d_nd_n^*a_p^{-1}\| = \lim_{p\to\infty} \|a_p^{-1}(d_nd_n^*)^{1/4}(d_nd_n^*)^{1/2}(d_nd_n^*)^{1/4}a_p^{-1}\| \\ &\leqq (\|d_nd_n^*\|^{1/4}+\delta)^2\|d_n\| \leqq \|d_n\|.\end{aligned}$$

Similarly $\|c_n\|^2 \leqq \|d_n\|$ and (2) holds.

LEMMA 2.5. *Let A be a B^*-algebra. The collection of all sets*

$$V_a = \{x \in M(A) : \|ax\| \leqq 1 \text{ and } \|xa\| \leqq 1\}$$

for $a \in A$ is a base at 0 in $M(A)$ for the strict topology.

Proof. The proof follows from a straightforward application of Lemma 2.4.

THEOREM 2.6. *Let A be a B^*-algebra and let $\{e_\lambda : \lambda \in \Lambda\}$ be an approximate identity for A. If H is a subset of $M(A)^*_\beta$, then the following statements are equivalent:*

(1) *H is β-equicontinuous.*

(2) *H is uniformly bounded and $e_\lambda \cdot f+f\cdot e_\lambda-e_\lambda\cdot f\cdot e_\lambda \to f$ uniformly on H, where $e_\lambda\cdot f(x)=f(xe_\lambda)$ and $f\cdot e_\lambda(x)=f(e_\lambda x)$ for all $x \in M(A)$.*

Proof. Assume (1) holds. Then H is contained in the polar of some basic neighborhood $V_a=\{x \in M(A) : \|ax\| \leqq 1$ and $\|xa\| \leqq 1\}$ of 0. Since the β-topology is weaker than the norm topology, it follows that H is uniformly bounded. Now for each $x \in M(A)$ and $\varepsilon > 0$ the element $x/(\|ax\|+\|xa\|+\varepsilon)$ belongs to V_a. So for $f \in H$

$$\begin{aligned} |f(x)| &= |(\|ax\|+\|xa\|+\varepsilon)f(x/(\|ax\|+\|xa\|+\varepsilon))| \\ &< \|ax\|+\|xa\|+\varepsilon. \end{aligned}$$

Since ε was picked arbitrarily, it follows that $|f(x)| \leqq \|ax\|+\|xa\|$. Hence

$$\begin{aligned} |(f-e_\lambda \cdot f - f\cdot e_\lambda + e_\lambda \cdot f \cdot e_\lambda)(x)| &= |f((1-e_\lambda)x(1-e_\lambda))| \\ &\leqq 2(\|ae_\lambda - a\| + \|e_\lambda a - a\|)\|x\| \end{aligned}$$

for each $f \in H$ and $x \in M(A)$. So for each $f \in H$

$$\|f-(e_\lambda \cdot f + f \cdot e_\lambda - e_\lambda \cdot f \cdot e_\lambda)\| \leqq 2\|ae_\lambda - a\| + 2\|e_\lambda a - a\|$$

and therefore it is clear that (2) holds.

Now assume (2) holds and that H is uniformly bounded by 1. To prove that H is β-equicontinuous, it will suffice to show that H is contained in the polar of some basic neighborhood of 0 in $M(A)^*_\beta$. For each $\lambda \in \Lambda$ set $R_\lambda f = e_\lambda \cdot f + f \cdot e_\lambda - e_\lambda \cdot f \cdot e_\lambda$ for each $f \in M(A)^*_\beta$ and set $S_\lambda x = e_\lambda x + x e_\lambda - e_\lambda x e_\lambda$ for each $x \in M(A)_\beta$. Now choose a sequence $\{e_{\lambda_n}\}$ of elements from our approximate identity such that for each positive integer n we have $\lambda_{n+1} > \lambda_n$, $\|R_{\lambda_{n+1}} f - R_{\lambda_n} f\| \leqq 1/4^{n+1}$ for each $f \in H$, $\|e_{\lambda_k} - e_{\lambda_k} e_{\lambda_{n+1}}\| \leqq 1/9 \cdot 4^n$ for $k = 1, 2, \ldots, n$, and $\|e_{\lambda_k} - e_{\lambda_{n+1}} e_{\lambda_k}\| < 1/9 \cdot 4^n$ for $k = 1, 2, \ldots, n$. Let $\{d_k\}$ be a sequence of elements in A defined by $d_{5k-4} = (3/2^{k+1}) e_{\lambda_k}$, $d_{5k-3} = e_{\lambda_k} - e_{\lambda_k} e_{\lambda_{k+1}}$, $d_{5k-2} = e_{\lambda_k} - e_{\lambda_{k+1}} e_{\lambda_k}$, $d_{5k-1} = e_{\lambda_k} - e_{\lambda_k} e_{\lambda_{k+2}}$, and $d_{5k} = e_{\lambda_k} - e_{\lambda_{k+2}} e_{\lambda_k}$. It is clear that $d_k \to 0$ uniformly and $\|d_k\| < 1$. Therefore, by Lemma 2.4, there exist sequences $\{b_k\}$ and $\{c_k\}$ of elements in A and a hermitian element $a \in A$, $\|a\| \leqq 1$, such that $d_k = ab_k = c_k a$ and $\max\{\|b_k\|^2, \|c_k\|^2\} \leqq \|d_k\|$. Set $a_1 = 8a$. We now wish to show that $H \subset V^0_{a_1}$, where $V^0_{a_1}$ is the polar of

$$V_{a_1} = \{x \in M(A) : \|a_1 x\| \leqq 1 \text{ and } \|x a_1\| \leqq 1\}$$

in $M(A)^*_\beta$. Since $d_{5k-4} = ab_{5k-4} = c_{5k-4} a$, we have for each $x \in V_{a_1}$ that $\|xe_{\lambda_k}\| = (2^{k+1}/3)\|xab_{5k-4}\| \leqq 2^{k+1}/3 \cdot 8$ and similarly $\|e_{\lambda_k} x\| \leqq 2^{k+1}/3 \cdot 8$. It follows, by straightforward computations, that for each $f \in H$ and $x \in V_{a_1}$ that

$$|R_{\lambda_k} f(x - S_{\lambda_{k+2}} x)| \leqq 1/2^{k+3}, \qquad |R_{\lambda_{k+1}} f(x - S_{\lambda_{k+2}} x)| \leqq 1/2^{k+3},$$

and

$$\|S_{\lambda_k} x\| \leqq 2^{k+1}/8.$$

These inequalities and the fact that $f = R_{\lambda_1} f + \sum_{k=1}^\infty (R_{\lambda_{k+1}} f - R_{\lambda_k} f)$ for each $f \in H$ imply that

$$|f(x)| \leqq |f(S_{\lambda_1}(x))| + \sum_{k=1}^\infty |(R_{\lambda_{k+1}} f - R_{\lambda_k} f)(x - S_{\lambda_{k+2}} x + S_{\lambda_{k+2}} x)| < 1$$

whenever $f \in H$ and $x \in V_{a_1}$. Hence $H \subset V^0_{a_1}$ and our proof is complete.

We will now generalize a result due to L. LeCam [11, Proposition 2, p. 217] and J. R. Dorroh [8] that concerns the β' or bounded strict topology. The β' topology is the strongest locally convex topology for $M(A)$ that agrees with the β topology on norm bounded sets. For a proof of existence, we refer the reader to [5] where an explicit neighborhood base is given. Another generalization of this theorem exists. F. D. Sentilles proved a similar result [15] in a Banach module setting and though we use the same technique his result does not seem to subsume our

COROLLARY 2.7. *If A is a B^*-algebra, then the β and β' topologies for $M(A)$ give the same dual. Consequently, $\beta=\beta'$.*

Proof. By virtue of Theorem 2.1, the proof that the β' dual of $M(A)$ is $M(A)^*_\beta$ is similar to the one given for Corollary 2.2. Therefore, it remains to be shown that $\beta=\beta'$. Let W be an absolutely convex β'-closed β'-neighborhood of 0. Then there exists a sequence $\{a_n\}$ of elements in A such that $B_n \cap V_{a_n} \subset B_n \cap W$, where $V_{a_n} =\{x \in M(A) : \|a_n x\| \leqq 1 \text{ and } \|x a_n\| \leqq 1\}$ and $B_n=\{x \in M(A) : \|x\| \leqq n\}$. Set $D_n = B_n \cap V_{a_n}$ and W' equal the β'-closed absolutely convex hull of $\bigcup D_n$. Then $W' \subset W$, and $(W')^0=\bigcap (D_n)^0$, where $(W')^0$ and $(D_n)^0$ are the polars of W' and D_n respectively in $M(A)^*_\beta$. We will show that $(W')^0$ is β-equicontinuous which implies that the β-closure of W' is a β-neighborhood. To this end, we will show that $e_\lambda \cdot f+f\cdot e_\lambda-e_\lambda\cdot f\cdot e_\lambda \to f$ uniformly on $(W')^0$, where $\{e_\lambda\}$ is an approximate identity for A consisting of positive elements. Let $\varepsilon>0$. Choose a positive integer n so that $1/n<\varepsilon$ and then choose a λ_0 so that for $\lambda \leqq \lambda_0$, $\|(1-e_\lambda)a_n\|<1/n$ and $\|a_n(1-e_\lambda)\| < 1/n$. Hence $\{n(1-e_\lambda)x(1-e_\lambda) : x \in B_1\} \subset D_n$ for $\lambda \geqq \lambda_0$. Therefore for $f \in (W')^0$, $x \in B_1$, and $\lambda \geqq \lambda_0$

$$|(f-e_\lambda\cdot f-f\cdot e_\lambda+e_\lambda\cdot f\cdot e_\lambda)(x)| \;=\; |f((1-e_\lambda)x(1-e_\lambda))| \;<\; 1/n \;<\; \varepsilon.$$

In other words, $\|f-e_\lambda\cdot f-f\cdot e_\lambda+e_\lambda\cdot f\cdot e_\lambda\|<\varepsilon$ for all $f\in(W')^0$ and $\lambda \geqq \lambda_0$. Thus, by Theorem 2.6, $(W')^0$ is a β-equicontinuous and our proof is complete.

It is well known that the bidual A^{**} of a B^*-algebra A is a W^*-algebra, and when A is canonically imbedded into A^{**}, A is a $*$-subalgebra of A^{**}. We will now consider the case when $M(A)$ is isometrically $*$-isomorphic to A^{**}. For example, if A is also an annihilator algebra, then this is true.

COROLLARY 2.8. *Let A be a B^*-algebra such that $M(A)$ is isometrically $*$-isomorphic to A^{**}. Then $M(A)_\beta$ is a Mackey space.*

Proof. The proof follows from Corollary 2.2, Corollary 2.3, Corollary 2.7, and [1, Theorem II.7, p. 292].

3. Proof of Theorem I and Theorem II.

LEMMA 3.1. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of B^*-algebras and let $A=(\sum A_\lambda)_0$. Then $M(A)$ is isometrically $*$-isomorphic to $\sum M(A_\lambda)$.*

Proof. Let $(R, S) \in M(A)$ and let $\lambda \in \Lambda$. Define R_λ and S_λ on A_λ by the formula $R_\lambda(a(\lambda)) = (R(a))(\lambda)$ and $S_\lambda(a(\lambda)) = (S(a))(\lambda)$ for each $a \in A$. To see that R_λ and S_λ are well defined, observe that if $a \in A$, with $a(\lambda) = 0$, and if $\{e_\alpha\}$ is an approximate identity for A, then by [4, Proposition 2.5, p. 80],

$$R(a)(\lambda) = \lim R(e_\alpha a)(\lambda) = \lim R(e_\alpha)(\lambda)a(\lambda) = 0,$$

and similarly, $S(a)(\lambda) = 0$. It is straightforward to show that $(R_\lambda, S_\lambda) \in M(A_\lambda)$ and that $\|(R_\lambda, S_\lambda)\| \leqq \|(R, S)\|$, so define the map $\mu: M(A) \to \sum M(A_\lambda)$ by the formula $\mu((R, S))(\lambda) = (R_\lambda, S_\lambda)$. It is clear that μ is a $*$-isomorphism from $M(A)$ into $\sum M(A_\lambda)$ and that $\|\mu((R, S))\| \leqq \|(R, S)\|$ for all $(R, S) \in M(A)$. Now for $(R, S) \in M(A)$ and $a \in A$, $\|a\| \leqq 1$,

$$\begin{aligned} \|R(a)\| &= \sup\{\|R(a)(\lambda)\| : \lambda \in \Lambda\} = \sup\{\|R_\lambda(a(\lambda))\| : \lambda \in \Lambda\} \\ &\leqq \sup\{\|R_\lambda\| : \lambda \in \Lambda\} = \sup\{\|(R_\lambda, S_\lambda)\| : \lambda \in \Lambda\} = \|\mu((R, S))\|. \end{aligned}$$

In other words, $\|(R, S)\| = \|\mu((R, S))\|$. Therefore to complete the proof we need to show that μ is onto. Let $\sum (R_\lambda, S_\lambda) \in \sum M(A_\lambda)$ and define $(R(a))(\lambda) = R_\lambda(a(\lambda))$ and $(S(a))(\lambda) = S_\lambda(a(\lambda))$ for each $a \in A$ and $\lambda \in \Lambda$. But it is clear that $(R, S) \in M(A)$ and $\mu((R, S)) = \sum (R_\lambda, S_\lambda)$. Hence μ is onto and our proof is complete.

LEMMA 3.2. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of B^*-algebras. Then the following statements are eqivalent*:

(1) *If $A = (\sum_{\lambda \in \Lambda} A_\lambda)_0$, then $M(A)_\beta$ is a Mackey space.*

(2) *If Λ_0 is a countable subset of Λ and $A_0 = (\sum_{\lambda \in \Lambda_0} A_\lambda)_0$, then $M(A_0)_\beta$ is a Mackey space.*

Proof. By virtue of Theorem 2.6, Lemma 3.1, and [10, p. 173], it is easy to show that (1) implies (2). Now let H be a β-weak* compact convex circled subset of $M(A)^*_\beta$ and let ϕ_λ denote the restriction map from $M(A)$ onto $M(A_\lambda)$, where $M(A_\lambda)$ is now viewed as a subspace of $M(A)$. Set $\Lambda_0 = \{\lambda \in \Lambda : \|\phi_\lambda f\| > 0$ for some $f \in H\}$. If Λ_0 is countable, then (2), together with Theorem 2.6, Lemma 3.1, and [10, p. 173], implies that H is β-equicontinuous and therefore, by [10, p. 173], (2) implies (1). Hence, it remains to be shown that Λ_0 is countable.

For each $\lambda \in \Lambda_0$ choose an $x_\lambda \in M(A_\lambda)$, $\|x_\lambda\| \leqq 1$, so that for some $f \in H$ we have $f(x_\lambda) \neq 0$. Now define $x \in M(A)$ by the formula

$$\begin{aligned} x(\lambda) &= x_\lambda \quad \text{if } \lambda \in \Lambda_0, \\ &= 0 \quad \text{if } \lambda \notin \Lambda_0, \end{aligned}$$

where we now view $M(A)$ as $\sum_{\lambda \in \Lambda} M(A_\lambda)$, and then define the map

$$T: C(\Lambda)_\beta \to M(A)_\beta$$

by the formula $T(\alpha)(\lambda) = \alpha(\lambda)x(\lambda)$ for each $\alpha \in C(\Lambda)$ and $\lambda \in \Lambda$. Here the topology for Λ is the discrete topology. Let $\{\alpha_i\}$ be a norm bounded net in $C(\Lambda)$ that converges to zero in the strict topology. It is straightforward to show that the net $\{T(\alpha_i)\}$ in $M(A)$ converges to zero in the strict topology and therefore, by virtue of Corollary 2.7, T is β-continuous. This implies that T has a well-defined adjoint map $T^*\colon M(A)^*_\beta \to C(\Lambda)^*_\beta$, which is continuous when both range and domain have their β-weak* topologies. It follows that $T^*(H)$ is β-weak* compact and therefore, by virtue of [6, Theorem 2.6, p. 478] and [6, Theorem 2.2, p. 476], Λ_0 is countable. Hence our proof is complete.

LEMMA 3.3. *Let A be a B^*-algebra and let a and b be hermitian elements in A such that $1 \geqq \|a\| \geqq \|b\|$. Then $\|a+b\| \leqq 1+2\|ab\|$.*

Proof. Let σ be the smallest number such that

$$\|c+d\| \leqq 1+\sigma \tag{3.1}$$

for all hermitian elements c and d in A, where $1 \geqq \|c\| \geqq \|d\|$ and $\|cd\| \leqq \|ab\|$. It is clear that such a number exists. Now if $\sigma > 2\|ab\|$, then $\|c+d\|^2 = \|(c+d)^2\| \leqq \|c^2+d^2\| + 2\|cd\| \leqq 1+\sigma+2\|ab\| < (1+\sigma)^2$ for all hermitian elements c and d in A, where $1 \geqq \|c\| \geqq \|d\|$ and $\|cd\| \leqq \|ab\|$. But this contradicts (3.1), so $\sigma \leqq 2\|ab\|$ and our proof is complete.

The author would like to thank Professor L. Eifler for the suggestion of the argument given for Lemma 3.3. This argument eliminated a longer proof.

REMARK. It follows immediately from Lemma 3.3 that for each pair of hermitian elements a, b in a B^*-algebra A the inequality $\|a+b\| \leqq \|a\| + 2\|ab\|/\|a\|$ holds whenever $\|a\| \geqq \|b\|$ and $\|a\| \neq 0$. In fact, there is a smallest number k such that $\|a+b\| \leqq \|a\| + k\|ab\|/\|a\|$ and this, in a sense, is a generalization of the triangle inequality for B^*-algebras. But $k=1$ when the B^*-algebra A is commutative, and this fact suggests the following question: *Is it true that $k=1$ only if A is commutative?*

Proof of Theorem I. Let $\{A_k\}_{k=1}^\infty$ be a sequence of B^*-algebras such that $M(A_k)_\beta$ is a Mackey space for each positive integer k. If we show that $M(A)_\beta$ is a Mackey space, where $A = (\sum_{k=1}^\infty A_k)_0$, then by virtue of Lemma 3.2 the proof will be complete. To this end, it will suffice to show that each β-weak* compact circled convex subset of $M(A)^*_\beta$ is β-equicontinuous. Now suppose that H is a β-weak* compact circled convex subset of $M(A)^*_\beta$ that is not β-equicontinuous. Since H is β-weak* compact, H is uniformly bounded and we can assume, without loss of generality, that H is uniformly bounded by 1. Let $\{e_\delta : \delta \in \Delta\}$ be an approximate identity for A consisting of positive elements. Then by virtue of Theorem 2.6 there exists an $\varepsilon > 0$ such that for each $\delta_0 \in \Delta$ we have

$$\|f - e_\delta \cdot f - f \cdot e_\delta + e_\delta \cdot f \cdot e_\delta\| \geqq 4\varepsilon \tag{3.2}$$

for some $f \in H$ and $\delta > \delta_0$. We will now define by induction a sequence of triples $\{(f_k, x_k, n_k)\}_{k=1}^\infty$ that satisfies the following conditions:

(1) $f_k \in H$, $x_k \in M(A)$, and n_k is a positive integer less than n_{k+1}.

(2) $\|x_k\| \leqq 1$, $x_k(q)=0$ for each positive integer $q \leqq n_{k-1}$ or $q > n_k$, where $M(A)$ is now viewed as $\sum_{k=1}^{\infty} M(A_k)$.

(3) $|f_k(x_k)| \geqq \varepsilon$.

By virtue of (3.2) there exists an f_1 in H, a δ in Δ, and a y in the unit ball of $M(A)$ such that $|f_1((1-e_\delta)y(1-e_\delta))| \geqq 3\varepsilon$. Since $M(A)_\beta^*$ under the strong topology is isometrically isomorphic to A^* and A^* is isometrically isomorphic to the L^1 direct sum of $\{A_k^*\}_{k=1}^{\infty}$, we can find a positive integer n_1 such that $|f_1(x_1)| \geqq \varepsilon$, where x_1 is the element in $M(A)$ defined by $x_1(q)=((1-e_\delta)y(1-e_\delta))(q)$ for $q=1, 2, \ldots, n_1$ and $x_1(q)=0$ for $q>n_1$. It is clear that (f_1, x_1, n_1) satisfies conditions (1), (2), and (3). Now assume that (f_k, x_k, n_k) has been defined for $k=1, 2, \ldots, p$. Let B_{n_p} be the subspace of A defined by $B_{n_p}=\sum_{k=1}^{n_p} A_k$ and let ϕ denote the restriction mapping from $M(A)$ to $M(B_{n_p})=\sum_{k=1}^{n_p} M(A_k)$. It is straightforward to show, by using Theorem 2.6, that $M(B_{n_p})_\beta$ is a Mackey space and therefore, by virtue of Theorem 2.6, [10, p. 173], and (3.2), there exists an f_{p+1} in H, a δ in Δ, and a y in the unit ball of $M(A)$ such that $|f_{p+1}((1-e_\delta)y(1-e_\delta))| \geqq 3\varepsilon$ and

$$(3.3) \qquad \|\phi(f_{p+1}-e_\delta \cdot f_{p+1}-f_{p+1}\cdot e_\delta+e_\delta \cdot f_{p+1}\cdot e_\delta)\| < \varepsilon.$$

By virtue of (3.3) and the fact that $M(A)_\beta^*$ under the strong topology is isometrically isomorphic to the L^1 direct sum of $\{A_k^*\}_{k=1}^{\infty}$, we can find a positive integer $n_{p+1} > n_p$ such that $|f(x_{p+1})| \geqq \varepsilon$, where x_{p+1} is the element in $M(A)$ defined by $x_{p+1}(q) =((1-e_\delta)y(1-e_\delta))(q)$ for $n_p < q \leqq n_{p+1}$ and $x_{p+1}(q)=0$ otherwise. It is clear that $(f_{p+1}, x_{p+1}, n_{p+1})$ satisfies conditions (1), (2), and (3), and our induction is complete. Now let x be the element in $M(A)$ defined by $x(q)=x_k(q)$ when $n_{k-1} < q \leqq n_k$. Then define the map $T\colon (l^\infty, \beta) \to M(A)_\beta$ by the formula $T(\alpha)(q)=\alpha(q)x(q)$ for each $\alpha \in l^\infty$ and positive integer q. By virtue of Corollary 2.7, it is straightforward to show that T is continuous. Hence T has a well-defined adjoint map $T^*\colon M(A)_\beta^* \to l^1$, which is continuous when both range and domain have the β-weak* topologies. Thus, $T^*(H)$ is a β-weak* compact subset of l^1 and this implies, by virtue of [6, Theorem 2.4, p. 477], that $T^*(H)$ is β-equicontinuous in l^1. Since $\sum_{k=1}^{q} \alpha(k)x(k)$ converges in the strict topology to $T(\alpha)$ as $q \to \infty$ for each $\alpha \in l^\infty$, we see that $T^*f(\alpha)=f(T(\alpha))=\sum_{k=1}^{\infty} \alpha(k)f(x(k))$ for each $f \in M(A)_\beta^*$ and $\alpha \in l^\infty$. So $T^*f=\{f(x(k))\}_{k=1}^{\infty}$. Since $T^*(H)$ is β-equicontinuous, there exists, by virtue of [6, Theorem 2.2, p. 476], a positive integer N such that $\sum_{k=N+1}^{\infty} |f(x(k))| < \varepsilon$ for each $f \in H$. This implies that $|f(x_q)| \leqq \sum_{k=n_q}^{n_{q+1}} |f(x(k))| < \varepsilon$ for $n_q > N$. This holds for all $f \in H$ and in particular $|f_q(x_q)| < \varepsilon$. But this contradicts (3). Hence H is β-equicontinuous and our proof is complete.

Proof of Theorem II. Let A be a B^*-algebra. If condition (1) holds, then it follows from Corollary 2.8 that $M(A)_\beta$ is a Mackey space. Now assume that A has a countable approximate identity. To show that $M(A)_\beta$ is a Mackey space it will suffice to show that every β-weak* compact subset of $M(A)_\beta^*$ is β-equicontinuous [10, p. 173]. Suppose that H is a β-weak* compact subset of $M(A)_\beta^*$ that is not β-equicontinuous. Since H is β-weak* compact, H is uniformly bounded, and without loss of generality we can assume that H is uniformly bounded by 1. Suppose $\{d_k\}_{k=1}^{\infty}$ is an approximate identity for A consisting of positive elements. We may assume that for each positive integer n

$$\|d_{n+1}d_k - d_k\| < 1/n \cdot 2^{n+3} \tag{3.4}$$

for $k=1, 2, \ldots, n$. Now because of Theorem 2.6 there exists an $\varepsilon > 0$ such that for each positive integer N the inequality

$$\|f - d_n \cdot f - f \cdot d_n + d_n \cdot f \cdot d_n\| \geqq 5\varepsilon \tag{3.5}$$

holds for some $f \in H$ and integer $n > N$. We will now define by induction a sequence of quadruples $\{(f_k, a_k, n_{2k-1}, n_{2k})\}_{k=1}^{\infty}$ that satisfies the following conditions:

(a) $f_k \in H$, a_k is a hermitian element in the unit ball of A, and n_{2k-1}, n_{2k} are positive integers such that $n_{2k-1} < n_{2k} < n_{2k+1}$.

(b) $|f_k(d_{n_{2k}}(1 - d_{n_{2k-1}})a_k(1 - d_{n_{2k-1}})d_{n_{2k}})| \geqq \varepsilon$.

By virtue of (3.5) and Corollary 2.3, it is straightforward to show that there exist an $f_1 \in H$, a hermitian element a_1 in the unit ball of A, and positive integers n_1, n_2 with $n_1 < n_2$ such that

$$|f_1(d_{n_2}(1 - d_{n_1})a_1(1 - d_{n_1})d_{n_2})| \geqq \varepsilon.$$

Thus (f_1, a_1, n_1, n_2) satisfies (a) and (b). Now suppose the quadruple $(f_k, a_k, n_{2k-1}, n_{2k})$ has been defined for $k=1, 2, \ldots, p$ so that conditions (a) and (b) have been satisfied. Again, by virtue of (3.5) and Corollary 2.3, it is straightforward to show that there exist an $f_{p+1} \in H$, a hermitian element a_{p+1} in the unit ball of A, and positive integers n_{2p+1}, n_{2p+2} with $n_{2p} < n_{2p+1} < n_{2p+2}$ such that

$$|f_{p+1}(d_{n_{2p+2}}(1 - d_{n_{2p+1}})a_{p+1}(1 - d_{n_{2p+1}})d_{n_{2p+2}})| \geqq \varepsilon$$

and our induction is complete. Set $x_k = d_{n_{2k}}(1 - d_{n_{2k-1}})a_k(1 - d_{n_{2k-1}})d_{n_{2k}}$ and $e_k = d_{2k}$. Because of (3.4), (a), and (b), $\{(f_k, x_k, e_k)\}_{k=1}^{\infty}$ is a sequence of triples such that the following conditions hold:

(a)′ $f_k \in H$, x_n is an hermitian element in the unit ball of A, and $e_k \in A$.

(b)′ $\{e_k\}$ is an approximate identity for A consisting of positive elements.

(c)′ For each positive integer p, $\|e_p x_k\| = \|x_k e_p\| < 1/2^k$ for $k = p+1, p+2, \ldots$ and $\|x_{p+1}x_k\| = \|x_k x_{p+1}\| < 1/p \cdot 2^{p+2}$ for $k=1, 2, \ldots, p$.

(d)′ $|f_k(x_k)| \geqq \varepsilon$.

Let $\alpha = \{\alpha_k\}_{k=1}^{\infty}$ belong to l^{∞}. By virtue of Lemma 3.3, it is straightforward to show that $\|\sum_{k=1}^{n} \alpha_k x_k\| \leqq \|\alpha\|_{\infty} \sum_{k=1}^{n} 1/2^{k-1} \leqq 2\|\alpha\|_{\infty}$ for each positive integer n. This inequality and the fact that $\|e_n x_p\| = \|x_p e_n\| < 1/2^p$ for $p \geqq n+1$, imply that the sequence of partial sums $\{\sum_{k=1}^{n} \alpha_k x_k\}_{n=1}^{\infty}$ is β-Cauchy. Since $M(A)_\beta$ is complete [4, Proposition 3.6, p. 83], we may define the map T: $(l^{\infty}, \beta) \to M(A)_\beta$ by the formula $T(\alpha) = \sum_{k=1}^{\infty} \alpha_k x_k$, where $\alpha = \{\alpha_k\}_{k=1}^{\infty}$ and $\sum_{k=1}^{\infty} \alpha_k x_k$ is the β-limit of the partial sums. By virtue of Corollary 2.7, it is straightforward to show that T is continuous and therefore T has a well-defined adjoint map T^*: $M(A)_\beta^* \to l^1$, which is continuous when both range and domain have the β-weak* topologies. Thus $T^*(H)$ is a β-weak* compact subset of l^1 and this implies, by virtue of [6, Theorem 2.4, p. 477], $T^*(H)$ is β-equicontinuous in l^1. Observe that $T^*f(\alpha) = f(T(\alpha)) = \sum_{k=1}^{\infty} \alpha_k f(x_k)$ for each $\alpha \in l^\infty$ and $f \in M(A)_\beta^*$, so that $T^*f = \{f(x_k)\}_{k=1}^{\infty}$. Since $T^*(H)$ is β-equicontinuous, there exists by virtue of [6, Theorem 2.2, p. 476] a positive integer N such that $\sum_{k=N+1}^{\infty} |f(x_k)| < \varepsilon$ for each $f \in H$. Thus, for $f \in H$ and $k > N$ we have $|f(x_k)| < \varepsilon$ and in particular $|f_k(x_k)| < \varepsilon$ for $k > N$. But this contradicts (d)′. Hence H is β-equicontinuous and our proof is complete.

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